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Hall effects for MHD Oldroyd 6-constant fluid flows using finite element method

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SUMMARY

This paper numerically investigates the influence of Hall current on the steady flows of an Oldroyd 6-constant fluid between concentric cylinders. The flow analysis has been performed by employing finite element method. Two flow problems are considered. These problems have been recently solved by Rana *et al.* (*Chaos, Solitons and Fractals*, in press). Here the main equation governing the flow problems in (*Chaos, Solitons and Fractals*, in press) is corrected first and then used in the simulation. Finally, the interesting observations are obtained by plotting graphs.

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KEY WORDS: finite element method; Oldroyd 6-constant fluid; steady flows; Hall currents; Poiseuille flow; Couette flow

1. INTRODUCTION

The non-Newtonian fluids are now acknowledged more appropriate in industrial and technological applications. In view of their importance, several authors [1–19] have recently studied non-Newtonian fluids in various situations. Very recently, Rana *et al.* [20] analyzed the magnetohydrodynamic (MHD) flows of an Oldroyd 6-constant fluid between two concentric cylinders. In fact, the conclusion that $m \to \infty$ gives the result of the hydrodynamic case is not correct. This happens due to a mistake in Equations (9), (12), (21), and (23) of Reference [20]. They discussed the Poiseuille and generalized Couette flows using finite difference scheme.

In the present paper, we first correct the equations and then reconsider the Poiseuille and generalized Couette flows. The finite element method is used in finding the numerical solution. The main points of the present analysis are included in the conclusions.

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2. PROBLEMS STATEMENT

We consider an incompressible and MHD Oldroyd 6-constant fluid between two coaxial cylinders of radii R_0 and R_1. The induced magnetic field is negligible and the Hall currents are considered. In order to avoid repetition, only the correct forms of Equations (9), (12), (21), and (23) are presented as follows:

$$\mathbf{J}\times\mathbf{B}=-\frac{\sigma B_0^2(1-\mathrm{i}m)}{1+m^2}\mathbf{V} \tag{1}$$

$$\frac{\partial p}{\partial z}=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rS_{rz})-\frac{\sigma B_0^2(1-\mathrm{i}m)}{1+m^2}u \tag{2}$$

$$\left[\frac{1+(3\alpha_1-\alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2+\alpha_1\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{\left(1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right)^2}\right]\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\left[\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}-\frac{\sigma B_0^2(1-\mathrm{i}m)}{\mu\left(1+m^2\right)}u=\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{3}$$

$$\left[\frac{1+(3\alpha_1-\alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2+\alpha_1\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{\left(1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right)^2}\right]\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\left[\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}-M^2\frac{(1-\mathrm{i}m)}{\left(1+m^2\right)}u=\frac{\mathrm{d}p}{\mathrm{d}z} \tag{4}$$

Note that the above equation is in dimensionless form. Here $\mathbf{J}$ is the current density, $\mathbf{B}$ is the total magnetic field, $\mathbf{B}_0$ is an applied magnetic field, σ is the electrical conductivity, m is the Hall parameter, p is the pressure, $\alpha_i\,(i=1,2)$ are material constants, u is the z-component of velocity $\mathbf{V}$, μ is the dynamic viscosity and

$$S_{rz}=\frac{\mu\frac{\mathrm{d}u}{\mathrm{d}r}+\mu\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2} \tag{5}$$

and $M^2=\sigma B_0^2/(\mu/R_0^2)$.

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3. IMPLEMENTATION OF FINITE ELEMENT METHOD

Equation (4) in associated variable form can be written as

$$\int_{\Omega}\phi\left(\left[\frac{1+(3\alpha_1-\alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2+\alpha_1\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{\left(1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right)^2}\right]\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\left[\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right]\frac{\mathrm{d}u}{\mathrm{d}r}-\frac{M^2}{1+m^2}u-\frac{\mathrm{d}p}{\mathrm{d}z}\right)\mathrm{d}r=0 \tag{6}$$

in which Ω is the domain of the problem, ϕ is arbitrary test functions that can be considered as the variation of u. We have chosen linear basis function for the solution of this equation, which is a two node line element. The finite element solution over this element is chosen of the following type:

$$\mathbf{u}=\sum_{i=1}^{2}u_i\psi_i \tag{7}$$

The Galerkin weighted residual method is applied for the solution of original equation. In Galerkin approach, the weight function for an element is chosen as

$$\phi=\psi_i \quad (i=1\dots2) \tag{8}$$

where ψ_i are the basis functions for a typical element (r_i, r_{i+1}) and are defined by

$$\psi_1=\frac{r_{i+1}-r}{r_{i+1}-r_i}, \quad \psi_2=\frac{r-r_i}{r_{i+1}-r_i} \quad r_i\leqslant r\leqslant r_{i+1} \tag{9}$$

Since Equation (6) is nonlinear in u, therefore a set of nonlinear algebraic equations is obtained when Equations (7) and (8) are invoked in Equation (6). In matrix form such set is written as

$$\mathbf{F}(\mathbf{u})=\begin{bmatrix} f_1(\mathbf{u})\\ f_2(\mathbf{u})\\ .\\ .\\ f_n(\mathbf{u})\end{bmatrix}=\mathbf{0} \tag{10}$$

where n is the total number of unknown solution values and $f_i(\mathbf{u})$ are obtained by putting in the value of $\mathbf{u}$ from Equation (7) for each element in Equation (6).

The domain of the problem is divided into a set of 200 elements of equal length. As we know the velocity values at the boundary of the domain, therefore after incorporating these boundary

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conditions we obtain a set of 199 simultaneous non linear algebraic equations having 199 unknowns. For the solution of this set of equations, we used Newton's iterative method with an initial guess provided to it as

$$u(r) = r \tag{11}$$

Newton's iterative method is generally implemented in two steps procedure. First vector $\mathbf{z}$ is found which will satisfy

$$J(\mathbf{u}^{(k)})\mathbf{z} = -\mathbf{F}(\mathbf{u}^{(k)}) \tag{12}$$

In the above equation, $J(\mathbf{u}^{(k)})$ is the Jacobian of $\mathbf{F}(\mathbf{u}^{(k)})$. The new approximation $\mathbf{u}^{(k+1)}$ can be obtained through

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \mathbf{z} \tag{13}$$

Newton's method is expected to give quadratic convergence, provided that a sufficiently good starting guess is chosen.

4. ANALYSIS OF RESULTS

4.1. Poiseuille flow

In this subsection, both cylinders are stationary and the flow is induced due to an applied constant pressure gradient. The dimensionless boundary conditions are [20]

$$\begin{aligned} u(r) &= 0 \quad \text{at } r = 1 \\ u(r) &= 0 \quad \text{at } r = k = R_1/R_0 \end{aligned} \tag{14}$$

Now Figures 1–4 have been prepared to display the influence of different parameters on the velocity. Note that in all figures except Figure 2 panel a stands for the Newtonian fluid and panel b for Oldroyd 6-constant fluid. The influence of constant pressure gradient on the velocity field is seen in Figure 1. It is noted that the velocity increases for both the Newtonian and Oldroyd 6-constant fluids when magnitude of constant pressure gradient is increased. It is also observed that

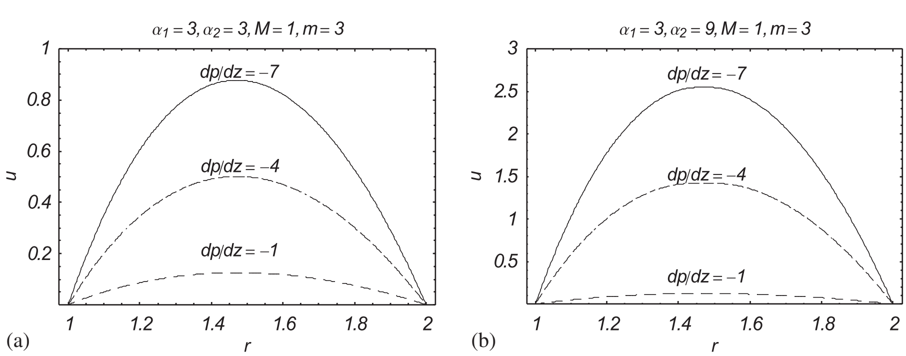

Figure 1. Influence of pressure gradient on u for Poiseuille flow.

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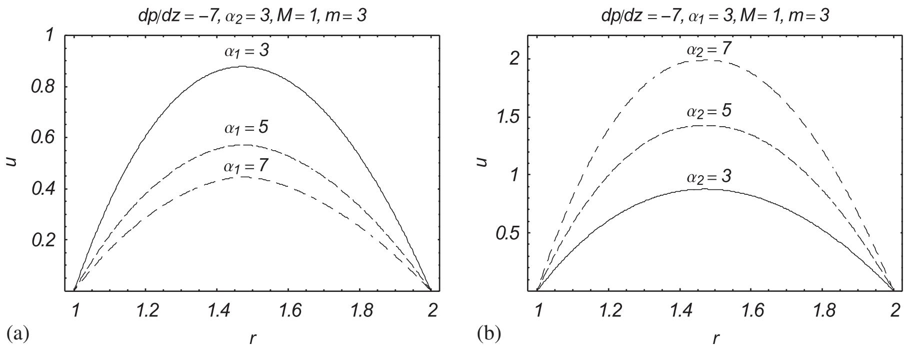

Figure 2. Influence of fluid parameters α_1 and α_2 on u for Poiseuille flow.

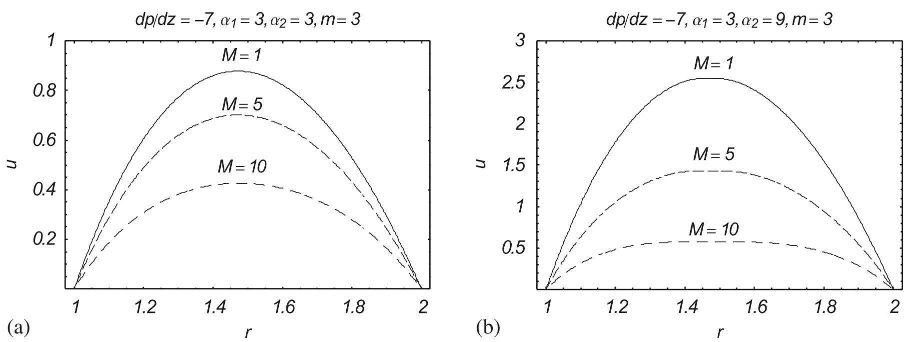

Figure 3. Influence of Hartmann number M on u for Poiseuille flow.

the non-Newtonian effects enhance the velocity. The influence of fluid parameters α_1 and α_2 on the velocity profiles is shown in Figure 2. Here the velocity decreases by increasing α_1 and fixed α_2. However, the effect of α_2 on the velocity is quite opposite to that of α_1. The velocity decreases by increasing the Hartman number M for both the Newtonian and Oldroyd 6-constant fluids and is depicted in Figure 3. The influence of Hall parameter on the velocity profile is shown in Figure 4. This figure elucidates that the velocity increases when Hall parameter is increased.

4.2. *Generalized Couette flow*

Here the flow is driven by the motion of inner cylinder and an applied pressure gradient. The boundary conditions in dimensionless form are [20]

$$\begin{aligned} u(r) &= 1 \quad \text{at } r = 1 \\ u(r) &= 0 \quad \text{at } r = k \end{aligned} \tag{15}$$

The variation of Hall parameter on the velocity field is given in Figure 5. It is observed that the velocity increases by increasing the Hall parameter. The increase in the magnitude of velocity is more in an Oldroyd 6-constant fluid when compared with the Newtonian fluid.

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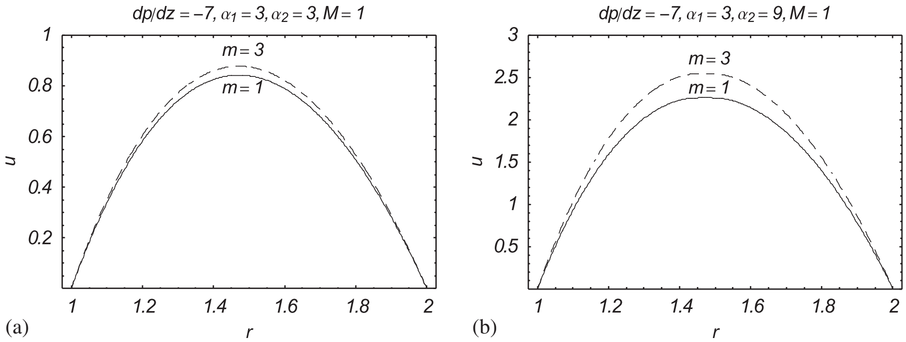

Figure 4. Influence of Hall parameter m on u for Poiseuille flow.

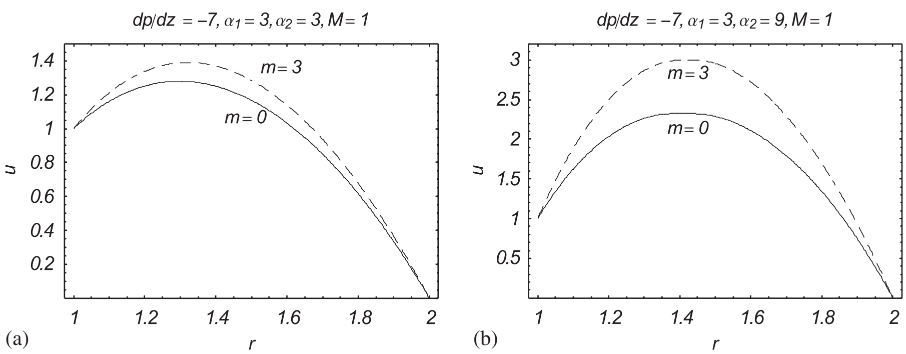

Figure 5. Influence of Hall parameter m on u for generalized Couette flow.

5. CONCLUSIONS

The main objective of this present study is to correct the main flow equation (4). Unlike the flow analysis of Reference [20] Equation (4) does not correspond to the hydrodynamic case when $m \to \infty$. Numerical simulation here is also achieved through finite element method. Reliability of the presented analysis is verified through the deduced observations. An extension to other features of the considered problems presents also the possibility of future work. Our major efforts in the future will be dedicated to the applications of the employed approach to different area.

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